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LETTER TO THE EDITOR

On the mean-field spin-glass instability at finite fields

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Abstract. We investigate the instability of the high-temperature phase of the Sherrington–Kirkpatrick model at finite magnetic fields. We find that the relevant function for this analysis is $q^{(12)}(h^{(1)}, h^{(2)}) \equiv \langle m_i(h^{(1)}) m_i(h^{(2)}) \rangle_{av}$ with two external fields $h^{(1)}$ and $h^{(2)}$ and reveal its non-analytic structure at the critical point. We indicate that the field breaking the 'hidden' symmetry of the high-temperature solution and causing a breakdown of linear response theory is $v \equiv h^{(1)} - h^{(2)}$.

The mean-field theory of spin glasses, being a solution of the long-range Ising model with random interactions, of Sherrington and Kirkpatrick (1975) (SK), is now understood to a great extent (for a recent review, see Binder and Young (1986)). It is generally accepted that the exact solution of the SK model yields the mean-field equations of Thouless *et al* (1977) (TAP) for one-bond configuration. These equations are taken as a starting point for a configurational averaging. At present, there are two successful ways of averaging the TAP equations at low temperatures: Parisi's replica symmetry breaking scheme (Parisi 1979, 1980a, b, c) and Sompolinsky's dynamical solution (Sompolinsky 1981, Sompolinsky and Zippelius 1981). Though initially invented within other approaches, it was afterwards shown that they both can be derived from the TAP equations after appropriate averaging (De Dominicis and Young 1983, Mézard *et al* 1986, Dasgupta and Sompolinsky 1983, Sommers *et al* 1983). However, they do not represent a direct way of averaging the TAP equations, since they use ansätze concerning a presumed low-temperature behaviour of spin glasses. These ansätze are different in Parisi's and Sompolinsky's approaches and are chosen in such a way as to obtain a marginally stable solution below the instability line of de Almeida and Thouless (1978) (AT). The additional information needed about the low-temperature behaviour of spin glasses is provided by numerical analysis of the SK model which says that this model and the TAP equations have many solutions at very low temperatures (Sherrington and Kirkpatrick 1978, Bray and Moore 1980b, De Dominicis *et al* 1980, Tanaka and Edwards 1980). We thus need infinitely-many-order parameters (e.g. $q(x)$, $x \in (0, 1)$) to determine the low-temperature spin-glass phase. Using this fact, we obtain a reasonable interpretation of the order-parameter function $q(x)$ of Parisi and Sompolinsky and a consistent picture of a complex many-valley phase space of spin glasses in infinite dimensions. Although equivalent in their main features, Parisi's and Sompolinsky's approaches differ in some aspects and predictions (cf Binder and Young 1986). We may thus ask what is the arbitrariness in the constructions of the low-temperature solutions of the SK model or, equivalently, whether there is an unambiguous way of averaging the TAP equations. Another question not yet satisfactorily explained is why the diagrammatic approach (Sommers 1978), representing the direct way of averaging the TAP equations, actually breaks down below the AT line and why it fails to meet the instability at finite fields. The breakdown of the diagrammatic averaging is only brought into connection with a breaking of linear-response theory (Bray and Moore 1980a, Sommers 1983). This is deduced from the divergence of the spin glass susceptibility χ_{SG} about which the diagrammatic solution does not yield any information. However, what the field is with respect to which linear response should be broken at finite magnetic fields, i.e. the symmetry breaking field, has not yet been specified.

In this letter, following the diagrammatic averaging of the TAP equations (Sommers 1978), we analyse the behaviour of the high-temperature solution of the SK model close to the AT line. We characterise the spin-glass transition by a symmetry breaking field and thus clarify the notion of the breakdown of linear response theory at finite fields and explain the failure of the SK solution to meet the spin-glass transition at finite fields.

The SK spin-glass model is described by a Hamiltonian

$$H = -\tfrac{1}{2}\sum_{ij} J_{ij}S_iS_j - \sum_i h_iS_i \tag{1}$$

with N Ising spins $S_i = \pm 1$ interacting via a set of exchanges J_{ij}. The J_{ij} are independent Gaussian random variables with

$$\langle J_{ij}\rangle_{\text{av}} = 0 \qquad \langle J_{ij}^2\rangle_{\text{av}} = J^2/N. \tag{2}$$

We shall assume in this letter only a non-zero homogeneous external magnetic field $h_i = h$. The mean-field equations of TAP determine magnetisations m_i:

$$m_i = \tanh(\beta h + \eta_i) \tag{3}$$

where

$$\eta_i = \sum_j \beta J_{ij}m_j - m_i \sum_{j\neq i} (\beta J_{ij})^2(1-m_j^2) \tag{4}$$

are local internal fields and $\beta = (k_BT)^{-1}$. Sommers (1978) showed that these fields are independent Gaussian random variables with

$$\langle \eta_i\rangle_{\text{av}} = 0 \qquad \langle \eta_i^2\rangle_{\text{av}} = Q \equiv (\beta J)^2\langle m_i^2\rangle_{\text{av}}. \tag{5}$$

The local fields η_i are functions of the external field h. We now shall assume a weak noise in the external field h and suppose an uncertainty in the determination of the dependence $m_i(h)$ in TAP equations (3). We are then to introduce an order parameter $q^{(12)}$ as a function of two magnetic fields $h^{(1)}$ and $h^{(2)}$:

$$q^{(12)}(h^{(1)}, h^{(2)}) \equiv \langle m_i(h^{(1)})m_i(h^{(2)})\rangle_{\text{av}} \tag{6}$$

where $m_i(h^{(1)})$ and $m_i(h^{(2)})$ are solutions of TAP equations (3) with external fields $h^{(1)}$ and $h^{(2)}$, respectively. Such a two-field function was introduced within the replica way (Blandin *et al* 1980, Parisi 1983), but its importance for the stability analysis of the high-temperature solution has not yet been revealed. However, it is almost immediately clear that $q^{(12)}(h^{(1)}, h^{(2)})$ is the relevant function to be studied since its second derivative is closely related to the spin-glass susceptibility

$$\chi_{SG}(h) = \frac{1}{N}\sum_{i,j}\left\{\left\langle\left(\frac{\partial m_i(h)}{\partial h_j}\right)^2\right\rangle_{\text{av}} - \left\langle\frac{\partial m_i(h)}{\partial h_j}\right\rangle_{\text{av}}^2\right\} \tag{7}$$

diverging at the transition to the spin-glass condensed phase.

We can evaluate the function $q^{(12)}(h^{(1)}, h^{(2)})$ analogously as $q_{EA}(h)$ was calculated in the Sommers paper with the only exception that we must distinguish two sets of internal fields $\eta_i^{(1)}$, $\eta_i^{(2)}$ being random Gaussian variables but not independent. Their overlap

$$\langle \eta_i^{(1)} \eta_j^{(2)} \rangle_{av} = \delta_{ij} Q^{(12)}(h^{(1)}, h^{(2)}) \equiv \delta_{ij} (\beta J)^2 q^{(12)}(h^{(1)}, h^{(2)})$$

is a new function in addition to $q^{(11)} = q_{EA}(h^{(1)})$, $q^{(22)} = q_{EA}(h^{(2)})$ which should be self-consistently determined. The set of self-consistent equations is obtained after a slightly modified Sommers procedure of averaging of the TAP equations. They are:

$$q^{(11)} = \int_{-\infty}^{\infty} \frac{d\bar{\eta}^{(1)}}{\sqrt{2\pi}} \exp\{-\bar{\eta}^{(1)^2}/2\} \tanh^2(\beta h^{(1)} + \sqrt{Q^{(11)}}\, \bar{\eta}^{(1)}) \tag{8a}$$

$$q^{(12)} = \int_{-\infty}^{\infty} \frac{d\bar{\eta}^{(1)}\, d\bar{\eta}^{(2)}}{2\pi} \exp\{-(\bar{\eta}^{(1)^2} + \bar{\eta}^{(2)^2})/2\} \tanh(\beta h^{(1)} + \sqrt{Q^{(11)}}\, \bar{\eta}^{(1)})$$
$$\times \tanh\left(\beta h^{(2)} + \frac{1}{\sqrt{Q^{(11)}}}(Q^{(12)} \bar{\eta}^{(1)} + \sqrt{\Delta}\, \bar{\eta}^{(2)})\right) \tag{8b}$$

where $\Delta \equiv Q^{(11)} Q^{(22)} - Q^{(12)^2}$. $q^{(22)}$ is obtained from ($8a$) by a simple interchange $1 \leftrightarrow 2$. The expression on the right-hand side of ($8b$) is not manifestly symmetric with respect to the interchange $h^{(1)} \leftrightarrow h^{(2)}$ since we have chosen a simpler diagonalisation of the matrix $Q^{(ab)}$ than the symmetric one. If the function $q^{(12)}$ is analytical, i.e. all derivatives are continuous, then

$$\lim_{h^{(2)} \to h^{(1)}} \frac{\partial^n}{\partial h^{(1)n}} q^{(12)} = \lim_{h^{(2)} \to h^{(1)}} \frac{\partial^n}{\partial h^{(2)n}} q^{(12)} = \frac{1}{2^n} \frac{\partial^n}{\partial h^{(1)n}} q^{(11)} \qquad n = 1, 2, \ldots \tag{9}$$

and $q^{(12)}$ reduces in this limit to $q^{(11)}$ of ($8a$). If (9) is violated, $q^{(12)}$ need not be identical with $q^{(11)}$ even at $h^{(1)} = h^{(2)}$ and the SK solution no longer has sense. We now show that the AT line can be defined as a line along which (9) is broken. We show that the first derivatives of $q^{(12)}$ are discontinuous and the second derivatives diverge at this line only if we approach the critical field $h^{(1)} = h^{(2)} = h_c(T)$ along any trajectory in $(h^{(1)}, h^{(2)})$ space different from the diagonal $h^{(1)} = h^{(2)}$ representing the SK solution.

To evaluate the first and second derivatives of $q^{(12)}$, we use some abbreviations, namely the brackets for the normalised integration over $\bar{\eta}^{(1)}$ and $\bar{\eta}^{(2)}$ from (8) and primes for derivatives with respect to either $h^{(1)}$ or $h^{(2)}$ when it is unambiguous. Otherwise, we add the subscript following the comma for specification of the derivative. We further denote

$$t_{1,2} \equiv \tanh x_{1,2}$$

$$x_1 = \beta h^{(1)} + \sqrt{Q^{(11)}}\, \bar{\eta}^{(1)} \qquad x_2 = \beta h^{(2)} + \frac{1}{\sqrt{Q^{(11)}}}(Q^{(12)} \bar{\eta}^{(1)} + \sqrt{\Delta}\, \bar{\eta}^{(2)}).$$

It is now simple to prove from (8) the variational formula

$$\delta\langle f_1 f_2 \rangle = \langle f_1' f_2 \rangle \delta h^{(1)} + \langle f_1 f_2' \rangle \delta h^{(2)} + \langle f_1'' f_2 \rangle \tfrac{1}{2} \delta Q^{(11)} + \langle f_1 f_2'' \rangle \tfrac{1}{2} \delta Q^{(22)} + \langle f_1' f_2' \rangle \delta Q^{(12)} \tag{10}$$

where f_1 and f_2 are arbitrary functions of x_1 and x_2, respectively. Formula (10) considerably simplifies the evaluation of the respective derivatives of $q^{(12)}(Q^{(12)})$. After applying (10) to (8) we obtain

$$[(\beta J)^{-2} - \langle t_1' t_2' \rangle] Q_{,1}^{(12)'} = \langle t_1' t_2 \rangle + \langle t_1'' t_2 \rangle \tfrac{1}{2} Q^{(11)'} \tag{11a}$$

$$[(\beta J)^{-2}-\langle t_1' t_2'\rangle]Q^{(12)''}_{,11}$$
$$=\langle t_1'' t_2\rangle(1+\tfrac{1}{2}Q^{(11)''})+\langle t_1''' t_2\rangle Q^{(11)'}+2\langle t_1'' t_2'\rangle Q^{(12)'}_{,1}$$
$$+\langle t_1''' t_2'\rangle Q^{(11)'}Q^{(12)'}_{,1}+\langle t_1'''' t_2\rangle(\tfrac{1}{2}Q^{(11)'})^2+\langle t_1'' t_2''\rangle(Q^{(12)'}_{,1})^2 \tag{11b}$$
$$[(\beta J)^{-2}-\langle t_1' t_2'\rangle]Q^{(12)''}_{,12}$$
$$=\langle t_1' t_2'\rangle+\langle t_1'' t_2'\rangle(\tfrac{1}{2}Q^{(11)'}+Q^{(12)'}_{,2})+\langle t_1' t_2''\rangle(\tfrac{1}{2}Q^{(22)'}+Q^{(12)'}_{,1})$$
$$+\langle t_1'' t_2''\rangle(\tfrac{1}{4}Q^{(11)'}Q^{(22)'}+Q^{(12)'}_{,1}Q^{(12)'}_{,2})+\langle t_1''' t_2'\rangle\tfrac{1}{2}Q^{(11)'}Q^{(12)'}_{,2}$$
$$+\langle t_1' t_2'''\rangle\tfrac{1}{2}Q^{(22)'}Q^{(12)'}_{,1}. \tag{11c}$$

For completeness, we add formulae for $Q^{(11)'}$ and $Q^{(11)''}$:

$$[(\beta J)^{-2}-\langle t_1 t_1''\rangle-\langle t_1'^2\rangle]Q^{(11)'}=2\langle t_1 t_1'\rangle \tag{12a}$$
$$[(\beta J)^{-2}-\langle t_1 t_1''\rangle-\langle t_1'^2\rangle]Q^{(11)''}$$
$$=2(\langle t_1'^2\rangle+\langle t_1'' t_1\rangle)+(\langle t_1 t_1'''\rangle+3\langle t_1' t_1''\rangle)Q^{(11)'}$$
$$+\tfrac{1}{2}[\langle t_1 t_1''''\rangle+4\langle t_1' t_1'''\rangle+3\langle t_1''^2\rangle]Q^{(11)'2} \tag{12b}$$

and analogously for $Q^{(12)'}_{,2}$, $Q^{(12)''}_{,22}$, $Q^{(22)'}$ and $Q^{(22)''}$. Inserting the expressions for the derivatives of $t\equiv\tanh$, i.e.

$$t'=1-t^2 \qquad t''=-2t(1-t^2) \qquad t'''=-2(1-t^2)(1-3t^2)$$
$$t''''=8t(1-t^2)(2-3t^2)$$

we would get explicit formulae for the derivatives, but they are rather lengthy and not interesting enough to be listed here.

Trivial manipulations of (11) and (12) are needed to prove that the limit in (9) holds true (at least up to $n=2$) only if

$$\beta^2 J^2\langle(1-t_1^2)^2\rangle\neq 1. \tag{13}$$

Stability of the solution, moreover, demands positivity of the LHS of (11). However, the most interesting situation is at the critical point $h_c(T)$, where (13) does not hold. We now expand $Q^{(12)}$ around this critical point in the high-temperature phase, where the solution of (8) is stable. We denote

$$u=h^{(1)}+h^{(2)}-2h_c(T)\geqslant 0 \qquad v=h^{(1)}-h^{(2)} \qquad |v|\leqslant u$$

and suppose that they are small, since $Q^{(12)}$ is continuous at $h^{(1)}=h^{(2)}=h_c(T)$. We then expand $Q^{(12)}$ from (8) around $h_c(T)$ to second powers of u, v and the variation of $Q^{(12)}$ denoted $\Delta Q^{(12)}$. We obtain a quadratic equation for $\Delta Q^{(12)}$. Its solution can be expressed in u, v and functions evaluated at $h_c(T)$, where $Q^{(12)}(h_c(T), h_c(T))=Q^{(11)}(h_c(T), h_c(T))=Q^{(22)}(h_c(T), h_c(T))=Q$. The resulting expression for $\Delta Q^{(12)}(u, v)$ is

$$\Delta Q^{(12)}(u,v)=\tfrac{1}{2}Q'u-\frac{1}{2\langle t''^2\rangle}(\langle\nabla t'^2\rangle u$$
$$\pm\{\langle\nabla t'^2\rangle^2u^2+2\langle t''^2\rangle[Q'(\langle\nabla t'^2\rangle+2\langle t't''\rangle)+2\langle t'^2\rangle]v^2\}^{1/2}). \tag{14}$$

We have used a condensed notation

$$\nabla t'^2\equiv 2t't''+(t't'''+t''^2)Q'$$

and

$$Q' \equiv \frac{\partial Q^{(12)}}{\partial u} = -\frac{2\langle tt' \rangle}{\langle tt'' \rangle}.$$

The brackets represent the integration from (8a) at $h_c(T)$. Stability demands the plus sign in (14) in the high-temperature phase. Equation (14) is the principal result of this letter. It displays the type of non-analyticity of the overlap function $Q^{(12)}(h^{(1)}, h^{(2)})$ at $h^{(1)} = h^{(2)} = h_c(T)$. We can now draw consequences from this formula. First, the SK solution is a special case of (14), namely $Q^{(12)}(u, 0)$. This function is analytical, as a function of one variable u, and no divergences appear at $h_c(T)$. The SK solution only changes the sign of the square root in (14) below the AT line. Thus, the failure of the diagrammatic approach (or the SK solution) to meet the spin-glass instability at $h_c(T)$ is because it uses only the variable u. Second, whenever we approach the critical point $h_c(T)$ along any trajectory with $v \neq 0$, all second derivatives of $Q^{(12)}$ diverge. Moreover, the values of the first derivatives of $Q^{(12)}$ at $h_c(T)$ depend on the way we approach the critical point. These derivatives are monotonous functions of the ratio $|v|/u$ and acquire the maximal absolute values at $|v|/u = 0$ (the SK solution) and the minimal ones for $|v|/u = 1$. Third, formula (14) enables us to determine a symmetry breaking field. This field can be characterised as a variable according to which some derivative of the free energy (or the order parameter) diverges at the critical point. Because

$$\left.\frac{\partial^2 \Delta Q^{(12)}(u, v)}{\partial v^2}\right|_{v=0}$$

diverges at $u = 0$, this symmetry breaking field can be identified with $v = h^{(1)} - h^{(2)}$. Notice that odd derivatives with respect to v vanish at $v = 0$, $u = 0$. Thus, the breaking of linear response theory in spin glasses means that below the AT line ($u < 0$) $Q^{(12)}$ as a function of v^2 admits another solution at $v = 0$ in addition to $Q^{(12)} = Q_{SK}$. This conclusion is in agreement with the existing low-temperature solutions (Parisi 1983, Binder and Young 1986). To describe the low-temperature phase, we must introduce explicitly the field v into the free energy and then perform the Legendre transformation to the new order parameter. Keeping $v = 0$, we have no tool (except for replicas) for the introduction of the low-temperature order parameter(s), analogously to the ferromagnet, if we do not break the global spin-flip symmetry. However, it is not clear whether the construction of the free energy as a function of v can be done unambiguously. The inability to define the values of the derivatives of $Q^{(12)}$ at $h_c(T)$ indicates problems with the multiplication of magnetisations (or internal fields η_i) at the same field h below the AT line. This is very reminiscent of point multiplication of generalised functions. We are then to regularise the multiplication of magnetisations, which leads to the introduction of infinitely many new (order) parameters. In the light of this, the present low-temperature constructions of the solution of the SK model can be viewed as regularisation schemes for point multiplication of generalised functions. The result of the previous reasoning, i.e. that the only order parameter conjugated to the symmetry breaking field v is not sufficient for a description of the low-temperature phase is supported by the fact that v does not fully regularise the solution below the AT line ($u < 0$). The physical limit $v \to 0$ cannot be simply performed even if we go beyond the linear response in v.

To summarise, we have shown that the quantity of principal importance for spin glasses is $q^{(12)}(h^{(1)}, h^{(2)})$ instead of the Edwards–Anderson parameter $q_{EA}(h)$. We have disclosed its non-analytic behaviour at the AT instability line and found a field breaking the 'hidden' symmetry of the high-temperature phase. This field then naturally explains the notion of the breakdown of linear response theory at finite fields and the failure of the SK solution to predict the instability. Knowledge of the symmetry breaking field enables us to deduce the type of spin-glass instability and the appearance of the low-temperature order parameters, entirely from the high-temperature behaviour close to the critical line, without using replicas and additional assumptions.

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